On the Poles of the Growth Series of Coxeter Groups

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Abstract : We present an overview of the problems connected with the number of real roots of the growth series of Coxeter groups.

Let (W,S) be a *Coxeter system* [B]. For any $T\subset S$, W_T denotes a subgroup generated by T and is called a *standard parabolic*. *The nerve* N_W of the Coxeter system (W,S) is the simplicial complex consisting of $T\subset S$ such that the subgroup W_T is finite.

Definition. *Assume that S is finite. A formal series $W(t)=\sum_{w\in W}t^{\ell(w)}$, where ℓ is a length function with respect to S, is called a* growth series *of W.*

The Euler Characteristic Conjecture. *Let M be a compact aspherical manifold of dimension $2n$, then The Euler characteristic satisfies*

$$(-1)^n\chi(M)\geq 0.$$

The construction of M. Davis [D] associates to (W,S) a contractible complex Σ_W together with a geometric (cocompact with finite stabilizers) action of W. If the nerve N_W of W is a simply connected generalized homology sphere (GHS) then Σ_W is a manifold. By the Selberg Lemma [Sel] one finds a torsion free subgroup Γ of finite index in W. Since every element of W, with nontrivial fixpoint set, is conjugated into some W_T of finite order [D], and thus torsion, the Euler Characteristic for $\Gamma\backslash\Sigma_W$ reads

Conjecture 1. *Assume that N_W is a (simply connected) GHS of dimension $(2n-1)$. Then*

$$\frac{(-1)^n}{W(1)}\geq 0.$$

Definition. *Define the f-polynomial f_X of a simplicial complex X by the formula*

$$f_X(t):=\sum_{\sigma\in X}t^{\#\sigma}.$$

Fact 2 [Ser, Prop. 26]. *$W(\cdot)$ is a series of a rational function. Moreover, if W is infinite, then*

$$\frac{1}{W(t^{-1})}=\sum_{T\subset S}\frac{(-1)^{\#T}}{W_T(t)},$$

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where T runs over subsets of S such that W_T is finite.

Corollary. *If the group W is right-angled then the growth series is a substitution of the f-polynomial of N_W:*

$$f_{N_W}\left(\frac{-t}{1+t}\right) = \frac{1}{W(t)}. \tag{3}$$

Proof : If W_T is finite, then $W_T(t) = (1+t)^{\#T}$, thus Fact 2 reduces to (3). □

Fact. *A simplicial complex X is a nerve of some right-angled Coxeter group if and only if it is* flag *i.e. for any subset T of the set of vertices, such that any two distinct elements of T are joined by an edge, T is a face of X.*

Thus Conjecture 1 implies

Charney-Davis Conjecture [CD2]. *If X is a flag GHS of dimension $(2n-1)$ then*

$$(-1)^n f_X(-1/2) \geq 0.$$

From the other hand Charney and Davis proved

Proposition (reciprocity of a growth series) [CD1]. *If the nerve of a Coxeter Group W is Eulerian (in particular GHS) of dimension $(n-1)$ then the growth series satisfies*

$$W(t) = (-1)^n W(t^{-1}).$$

If W is right-angled the above property is known as Dehn-Sommervile relations [K]. In this case the conjecture of Charney and Davis implies that the number of real poles of the growth series of W in the interval $(0,1)$ (assume for simplicity that 1 is not a pole) has the same parity as half of the number of all of the poles. Since the f-polynomial cannot have real positive zeroes (f-polynomial by definition has positive coefficients), by reciprocity, the number of the poles of the growth series of W in the interval $(0,1)$ is equal to the half of the number of all real poles. This leads to the following

Question. *Assume that X is a flag simplicial GHS. Are all the zeroes of f_X real?*

For relations of the above question to combinatorics and representation theory we refer the reader to to [RW] and [DDJO] respectively. The case $n \leq 6$ is discussed in detail in [G2].

Let $(n-1)$ be a dimension of a flag simplicial GHS. It is easy to observe that flag GHS of dimension $(n-1)$ has only real roots if $n = 2$ or 3. The stronger result of [DO] implies the same for $n = 4$. In [G1] we extend this result to $n = 5$.

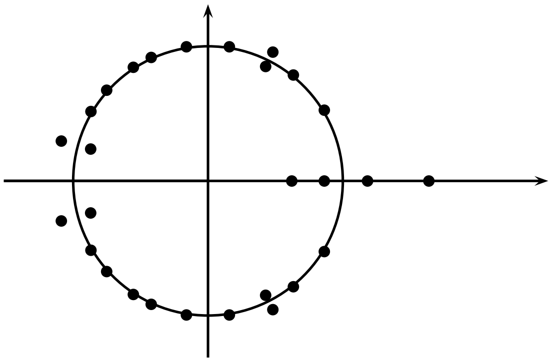
In [G1] and [G2] we construct right-angled Coxeter groups that fail having only real roots. They can be obtained as boundary complexes of convex polytopes starting from $n = 6$.

Since the zeroes of f-polynomial corespond to the poles of the growth series of the corresponding right-angled group, one may ask what is the nature of the poles of an arbitrary Coxeter group such that its nerve is a GHS. For example how many real poles can it have (note that the degree of the denominator of the growth series of non right-angled Coxeter group may be arbitrary greater then the dimension its nerve).

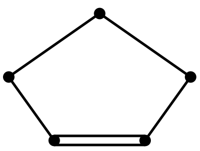
Below we present, what is known about this topic.

If W is affine Coxeter group then there is a unique real pole of order n at 1 [B]. If the dimension of a nerve $(n-1)$ satisfies $n \leq 3$ then there are exactly n positive real roots of W [P]. Moreover in those two cases all the non-real poles lie on the unit circle.

Usually if the nerve is a GHS of dimension at least 4 the non-real poles of the growth series fail to lie on the unit circle. Typical[1)] picture looks as follows:



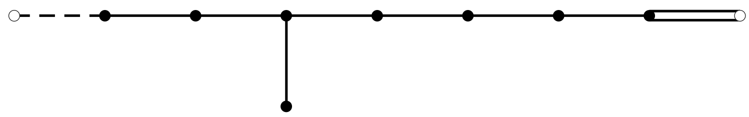
The above picture shows the poles o the growth series of the Coxeter group with the following Dynkin diagram:



In know examples, n poles seem to lie "near" the real positive half-line and the rest "near" the unit circle. However this observation may be only due to low (combinatorial) complication of nerves of tested groups.

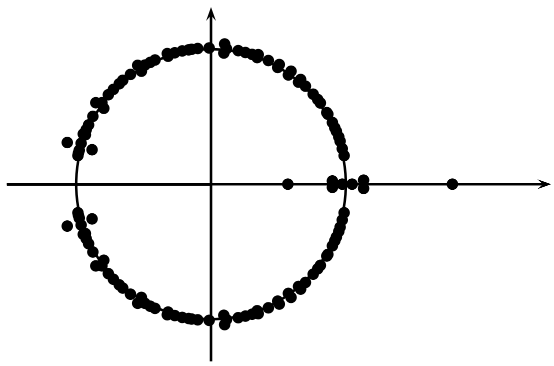
Let us focus on the construction from [G1]. The right-angled Coxeter group $\overline{W_T}$, is obtained as a normal closure of a certain (infinite) parabolics W_T in a bigger Coxeter group W. In the particular example the group W is defined by the following Dynkin diagram

1) We have testet a number groups, such that the nerve is a simplex or a product of symplicies.

(4) 

where T consist of white dots.

The following plot shows the poles of $W(\cdot)$ for W defined by the Diagram 4:



In [G1] we discuss connection between the growth series of W and $\overline{W_T}$. Both of them are specifications of the multi-variable growth series of the bigger group W. Unfortunately it does not allow to deduce the deficiency (less real poles of a growth series than predicts the dimension of the nerve) of one of the groups from the deficiency of another.

It is unknown if there any (necessary non right-angled) Coxeter group with deficiency for $n = 4$ or 5.

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