Scrapnote on Morse Inequalities

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Let h be h-vector of a (convex) simple flag polytope. The conjectures of Januszkiewicz say:

(J1) The zeroes of h are all real (and negative of course),

(J2) The cohomologies H_t of a weighted Davis complex of right-angeld Coxeter group with temperature t are concentrated in dimention $\kappa(t) = \#\{0 < \tau < t | h(-\tau) = 0\}$.

The Poincaré series of a chain complex that gives H_t is equal to $h(q(1+t)+1)$. (J2) implies

Conj-rollary (Morse inequalities).

$$\frac{h((q(1+t)+1) - (-q)^{\kappa(t)}h(-t)}{1+q} \tag{1}$$

has positive coeffitients at the powers of q.

We will prove a version of the above not relaing on the above cojectures:

Proposition. *If* h *is any polynomial with positive coefficients and* $0 < t \leq 1$ *(*unfortunately not suitable for hep-th*), then* (1) *has positive coeffitients at the powers of* q.

The proof relies on two lemmas:

Lemma 1.

$$(q+1)\sum_{k=1}^{N}\int_{-t}^{1}\frac{(q(s+t))^{k-1}}{(k-1)!}h^{(k)}(s)ds = h(q(1+t)+1) - h(-t).$$

Lemma 2. *Assume* $\kappa(t) > k$. *There exist* $t > t_k > 0$ *such that*

$$(-1)^k h(-t) < \int_{-t}^{-t_k}\frac{(s+t)^k}{k!}h^{(k+1)}(s)ds.$$

Proof of Proposition :

$$\begin{aligned}\frac{h((q(1+t)+1) - (-q)^{\kappa(t)}h(-t)}{1+q}& \\ &= \frac{h((1+q)(1+t)-t) - h(-t)}{1+q} - h(-t)\frac{1-(-q)^{\kappa(t)}}{1+q} \\ &= \sum_{k=0}^{N-1} q^k \int_{-t}^{1}\frac{(s+t)^k}{k!}h^{(k+1)}(s)ds - h(-t)\sum_{k=0}^{\kappa(t)-1}(-q)^k\end{aligned}$$

The coeficient at q^k is greater or equal to $\int_{-\tau}^{1} \frac{(s+t)^k}{k!} h^{(k+1)}(s)ds$, where $\tau = \begin{cases} t & \text{if } k \geq \kappa(t), \\ t_k & \text{otherwise.} \end{cases}$

One observes that $h(s) > h(-s)$ and $t + s > t - s$ for $s > 0$, so $\int_{-\tau}^{\tau} \frac{(s+t)^k}{k!} h^{(k+1)}(s)ds > 0$. $\square$

Proof of Lemma 1:

$$\begin{aligned}
(q+1)\sum_{k=1}^{N}\int_{-t}^{1}\frac{(q(s+t))^{k-1}}{(k-1)!}h^{(k)}(s)ds &= \sum_{k=1}^{N}\int_{-t}^{1}\frac{(q(s+t))^{k-1}}{(k-1)!}h^{(k)}(s)ds \\
&+ \sum_{k=1}^{N}\left(h^{(k)}(1)\frac{(q(1+t))^k}{k!} - \int_{-t}^{1}\frac{(q(s+t))^k}{k!}h^{(k+1)}(s)ds\right) \\
&= \int_{-t}^{1} h'(s)ds + \sum_{k=1}^{N} h^k(1)\frac{(q(1+t))^k}{k!} \\
&= h(q(1+t)+1) - h(-t). \quad \square
\end{aligned}$$

Proof of Lemma 2:

Let $-t_0$ be the $(k+1)$th root of h (couning from 0 to the left). By asumption $t_0 < t$. We have also $(-1)^{k-1}h'(t_0) \geq 0$.

Inductively, let $-t_i$ be the smallest ($(k+1-i)$th if one asumes (J1)) root of $h^{(i)}$ such that $t_i \leq t_{i-1}$. Then $(-1)^{k-i}h^{(i)}$ is positive on $(-t_{i-1}, t_i)$.

Between any pair of zeroes of a function there are some zeroes of its derivative, therefore there are at least $k+1-i$ zeroes of $h^{(i)}$ in $[-t_{i-1}, 0)$.

Then

$$\begin{aligned}
(-1)^k h(-t) =& (-1)^{k-1}(h(-t_0) - h(-t)) \\
=& (-1)^{k-1}\int_{-t}^{-t_0} h'(s)ds < (-1)^{k-1}\int_{-t}^{-t_1} h'(s)ds \\
& (\textit{integration by parts}) \\
=& (-1)^{k-2}\int_{-t}^{-t_1}(t+s)h''(s)ds < \ldots < \int_{-t}^{-t_k}\frac{(s+t)^k}{k!}h^{(k+1)}(s)ds. \quad \square
\end{aligned}$$